Indian Statistical Institute, Bangalore
M. Math II, Second Semester, 2022-23
Final Examination, Quantitative Finance

20.04.23 **Maximum Score 100** **Duration: 3 Hours**

Students are allowed to bring 6 pages (one-sided A4) of notes.

1. (12+3) Use Ito formula to show that

$$E(W(t)^k) = \frac{k(k-1)}{2}\int_0^t E(W(s)^{k-2})ds.$$

Use this to find $E(Z^8)$ where $Z \sim \mathcal{N}(0,1)$.

2. (10+5+10) For the Black-Scholes model under risk neutral measure with $r > 0$, we obtained the price of an up and in barrier option with payoff 1 and barrier at $A(> S_0)$ as

$$e^{-\theta^2/2}\int_0^\infty e^{-\theta\alpha}(e^{\theta x} + e^{-\theta x})e^{-(x+\alpha)^2/2T}\frac{1}{\sqrt{2\pi T}}dx.$$

Here $\theta = -\frac{1}{\sigma}(r - \frac{\sigma^2}{2})$ and $\alpha = \frac{1}{\sigma}\ln(A/S_0)$.

   (a) Express the price in terms of the standard normal cdf.
   (b) Show that the event $M_T > A$ is identical to the event $\tau_A < T$, where M_T is the maximum of the share price of the stock over the interval $[0,T]$ and τ_A is the first time that the share price of the stock reaches A.
   (c) Find the price of an option with payoff $e^{-\beta\tau_A}$, for some fixed β.

3. (15) Suppose a portfolio is made up of n assets. Suppose also that the price of the i-th asset, say $X_i, i = 1, 2, \cdots, n$ is an exponential random variable with parameter λ_i and these prices are independent of each other. Then $Y = \max(X_1, \cdots, X_n)$ will be the price of the most expensive asset. Find the value at risk and expected shortfall of Y at level α.

4. (8+7+5+5) There are k risky assets in a market and their returns have a mean vector μ and a positive definite covariance matrix Σ.

   (a) Find the weight vector w such that the return of a portfolio with this weight has a mean μ_b and lowest possible variance. Note that the sum of weights is 1. Also μ, Σ and μ_b are all known.
   (b) Show that the variance (σ_w^2) of the return of this portfolio is a quadratic function of μ_b.
   (c) Conclude from here that the efficient frontier is concave.
   (d) Suppose there is a risk-free instrument with return r_f. Consider all portfolios based on the k risky assets and the risk-free instrument. For a fixed level of standard deviation (s), the highest mean that can be attained by any such portfolio is m. Show that the point (s, m) lies on the line tangent to the efficient frontier passing through the point $(0, r_f)$.

5. (5+5+5+5) Consider a market with only two risky assets, A and B, and a risk-free asset. Stock A has 200 shares outstanding, a price per share of \$3.00, an expected return of 16% and a volatility of 30%. Stock B has 300 shares outstanding, a price per share of \$4.00, an expected return of 10% and a volatility of 15%. The correlation coefficient $\rho_{AB} = 0.4$.

   (a) What is expected return of the market portfolio?
   (b) What is volatility of the market portfolio?
   (c) What is the beta of each stock under the Capital Asset Pricing Model (CAPM)?
   (d) Does there exist a risk free rate for which the CAPM equation holds for both assets? Justify your answer.